

LOOP SPACES AS COMPLEX MANIFOLDS

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1. Introduction

Given a topological space M , its loop space consists of mappings of the circle S^1 into M . Depending on what conditions we impose on the mappings we get several loop spaces associated with M . If M has more structure than just topological, the loop spaces tend to inherit this structure: for example if M is a (Riemannian) manifold, the space of smooth loops is also a (Riemannian) manifold, albeit infinite dimensional. There is nothing surprising about this. In some cases, however, it happens that the *interaction* of the structures of M and S^1 gives rise to a structure on a loop space. For example with G a compact Lie group, the space of smooth loops in G modulo the action of G is a complex manifold (see [20]). Similarly, the manifold $\text{Diff } S^1/S^1$ is also a complex manifold (see [3], [10]). Here $\text{Diff } S^1$ stands for the space of smooth, orientation preserving diffeomorphisms of the circle, hence can be thought of as a space of embedded smooth loops in S^1 .

More recently J. Brylinski observed that the manifold of smooth, oriented, unparametrized knots in an oriented Riemannian manifold (M, g) of dimension 3 also has a complex structure; see [4]. We shall now describe this complex structure, which naturally lives on the space of immersed rather than embedded loops (knots).

Thus, let \mathfrak{M} denote the set of equivalence classes of smooth (meaning C^∞) immersions $f: S^1 \rightarrow M$. Two immersions $f_1, f_2: S^1 \rightarrow M$ are equivalent if $f_1 = f_2 \circ \varphi$, with φ an orientation preserving diffeomorphism of S^1 . Elements of \mathfrak{M} are called immersed loops. First we endow \mathfrak{M} with a topology as follows. Fix an immersed loop $\Gamma \in \mathfrak{M}$ represented by $f: S^1 \rightarrow M$. Let $\nu \rightarrow S^1$ denote the normal bundle of f :

$$\nu = \bigcup_{t \in S^1} \{v \in T_{f(t)}M : v \perp f_* T_t S^1\},$$

and exp the (partially defined) exponential map $\nu \rightarrow M$. ν inherits a Riemannian metric and a connection from TM , and so it makes sense

to speak of C^k -norms $|s|_k$ of its sections $s \in C^\infty(\nu)$ ($k = 1, 2, \dots$). Given now a small positive ϵ and a positive integer k , put

$$(1.1) \quad \mathfrak{U}(f, k, \epsilon) = \{[\exp \circ s] : s \in C^\infty(\nu), |s|_k < \epsilon\},$$

where $[g]$ denotes the class of g . By declaring the sets $\mathfrak{U}(f, k, \epsilon)$ a neighborhood basis of Γ we obtain a topology on \mathfrak{M} .

In fact, since $\mathfrak{U}(f, k, \epsilon)$ can be identified with open subsets in Fréchet spaces (e.g. in the space of smooth mappings $S^1 \rightarrow \mathbb{R}^2$), we obtain a Fréchet manifold structure on \mathfrak{M} . The tangent space $T_\Gamma \mathfrak{M}$ can be identified with $C^\infty(\nu)$. Define an endomorphism $\mathfrak{J} : T_\Gamma \mathfrak{M} \rightarrow T_\Gamma \mathfrak{M}$ by $\mathfrak{J}s_1 = s_2$ if for every $t \in S^1$, $s_1(t)$ and $s_2(t)$ are orthogonal and have the same length, and their vector product $s_1(t) \times s_2(t) \in f_* T_t S^1$ points in the direction of the orientation of Γ . Then $\mathfrak{J}^2 = -\text{id}$, and so \mathfrak{J} defines an almost complex structure on \mathfrak{M} (see [16]). The complexified tangent bundle $\mathbb{C} \otimes T\mathfrak{M}$ splits as $T^{1,0}\mathfrak{M} \oplus T^{0,1}\mathfrak{M}$, with $T^{1,0}\mathfrak{M}$ ($T^{0,1}\mathfrak{M}$) the eigenspaces of \mathfrak{J} corresponding to the eigenvalue i (resp. $-i$). Brylinski proves that \mathfrak{J} is formally integrable, that is for sections $\mathfrak{X}, \mathfrak{Y}$ of $T^{1,0}\mathfrak{M}$ $[\mathfrak{X}, \mathfrak{Y}]$ is again a section of $T^{1,0}\mathfrak{M}$. (Strictly speaking, [4] proves integrability only on the subset of \mathfrak{M} consisting of [singular] knots, but this subset being dense, integrability on \mathfrak{M} follows.)

Now the question arises whether this formally integrable almost complex structure is locally integrable. As a matter of fact, local integrability can be understood in several different ways. In this context the most natural (and most restrictive) concept is that of a bona fide complex manifold. This asks for the existence of a neighborhood \mathfrak{U} of an arbitrary point $\Gamma \in \mathfrak{M}$ and a holomorphic C^1 diffeomorphism (biholomorphism) $F : \mathfrak{U} \rightarrow V$ with V some open subset of a Fréchet space over \mathbb{C} . Here a C^1 mapping F_0 of an open subset of \mathfrak{M} into a Fréchet space is holomorphic if any local section \mathfrak{X} of $T^{0,1}\mathfrak{M}$ annihilates it: $\mathfrak{X}F_0 = 0$.

On the other end of the scale, one can consider the following weak notion of local integrability (as in [4]): given any $\Gamma \in \mathfrak{M}$ and $\mathfrak{X} \in T_\Gamma \mathfrak{M}$, $\mathfrak{X} \neq 0$, there is a neighborhood $\mathfrak{U} \subset \mathfrak{M}$ of Γ and a holomorphic function $F : \mathfrak{U} \rightarrow \mathbb{C}$ such that $\mathfrak{X}F \neq 0$.

In finite dimensions formal integrability and the above versions of local integrability are all equivalent, the difficult implication being the content of the Newlander-Nirenberg Theorem; see [18]. On Fréchet manifolds, where even real vector fields may fail to be integrable, the Newlander-Nirenberg Theorem does not hold. This leaves the question of local integrability of the complex structure of \mathfrak{M} open.

Momentarily we shall abandon \mathfrak{M} for another type of loop space. Independently of Brylinski's work, in 1988–89 I observed that certain loop spaces associated with Cauchy-Riemann (CR for short) manifolds (for definitions see §§2,3) also carry formally integrable almost complex structures. I hoped to use these structures to study the tangential Cauchy-Riemann equations; however, S. Baouendi pointed out that in my approach I was tacitly assuming that the structures in question are locally integrable in the strong (complex manifold) sense.

It soon turned out that this assumption is untenable and, indeed, the loop spaces I considered are generally speaking not locally biholomorphic to open subsets of Fréchet spaces. The proof, which will be given in §5 (see also §6), uses a theorem of Hans Lewy about analytic continuation of solutions of the tangential Cauchy-Riemann equations. But here is a pleasant twist! The careful reader of Lewy's paper will notice that his proof, in turn, revolves around the complex structure of CR loop spaces, even though the space of all loops never actually appears there (let alone the complex structure). We shall say a little more about that in §3.

Returning to Brylinski's loop space \mathfrak{M} , we will see that it can be holomorphically embedded into some CR loop space \mathfrak{N} . The construction of the embedding uses ideas from twistor theory, as advanced by LeBrun in [13]. That twistor theory should be of use in the study of \mathfrak{M} was first observed by Drinfeld and LeBrun, who, however, restricted their attentions to spaces of real analytic loops. Anyway, we can use the embedding to deduce that \mathfrak{M} is not locally biholomorphic to open subsets of Fréchet spaces, either (see §10).

On the other hand, the almost complex structure of both types of loop spaces can be shown to be locally integrable in the weak sense, when the CR manifold in question is embeddable in a complex manifold or when the Riemannian manifold is real analytic. For the loop spaces associated with CR manifolds this is quite straightforward (see §4); for \mathfrak{M} it will then follow since it embeds in a CR loop space; see §9.

2. Cauchy-Riemann manifolds

Let us start with a complex manifold Q of finite dimensions $n \geq 2$, and $N \subset Q$ a piece of a smooth real hypersurface (of real dimension $2n - 1$). The complexified tangent bundle $\mathbb{C} \otimes TQ$ splits as $T^{1,0}Q \oplus T^{0,1}Q$; in local coordinates $T^{1,0}Q$ ($T^{0,1}Q$) is spanned by $\partial/\partial z_j$ (resp. $\partial/\partial \bar{z}_j$).

Put

$$H^{1,0}N = T^{1,0}Q|_N \cap (\mathbb{C} \otimes TN); \quad H^{0,1}N = T^{0,1}Q|_N \cap (\mathbb{C} \otimes TN).$$

Then we have $\overline{H^{1,0}N} = H^{0,1}N$, $H^{1,0}N \cap H^{0,1}N = (0)$, and

$$(2.1) \quad [H^{1,0}N, H^{1,0}N] \subset H^{1,0}N,$$

meaning that the Lie bracket of any two sections of $H^{1,0}N$ is again a section of $H^{1,0}N$.

In general, a smooth manifold N of odd dimension $2n - 1$ equipped with two smooth subbundles $H^{1,0}N, H^{0,1}N \subset \mathbb{C} \otimes TN$ of rank $n - 1$ is called a Cauchy-Riemann manifold if the bundles $H^{1,0}N, H^{0,1}N$ have the properties listed above. Of course, to define a CR manifold, it suffices to specify $H^{1,0}N$ or $H^{0,1}N$, and the other bundle is then determined. Thus hypersurfaces in complex manifolds are CR manifolds (but not every CR manifold can be embedded into a complex manifold).

A C^1 function $u: N \rightarrow \mathbb{C}$ is a CR function if

$$(2.2) \quad Xu = 0 \quad \text{whenever } X \in H^{0,1}N.$$

Equations (2.2) are called tangential Cauchy-Riemann equations. When N is embedded in a complex manifold Q , traces of holomorphic functions defined on Q are examples of CR functions on $N \subset Q$. Lewy's theorem is a converse to this; it applies when N is not very degenerate in the following sense:

Definition 2.1. A point $p \in N$ is said to be *Levi flat* if for any smooth section X of $H^{1,0}N$, defined near p ,

$$(2.3) \quad [X, \bar{X}](p) \in H_p^{1,0}N \oplus H_p^{0,1}N.$$

We remark here that whether for a given X (2.3) holds or not depends only on $(N$ and) the value of X at p .

This definition applies to abstractly defined CR manifolds, but let us now assume that N is a hypersurface in a complex manifold Q .

Lewy's Theorem (See [14],[22]). *Any non-Levi flat point $p \in N$ has a neighborhood basis consisting of open sets $G \subset Q$ such that $G \setminus N$ has two components and one of them, say G_1 , has the following property. Given a CR function u on $G \cap N$ there is a function $\tilde{u} \in C^1(G_1 \cup (G \cap N))$, holomorphic on G_1 , which agrees with u on $G \cap N$.*

3. Loop spaces associated with CR manifolds

Let N be a CR manifold as in §2, and with notation as there, put $HN = TN \cap (H^{1,0}N \oplus H^{0,1}N)$. This is a real vector bundle of rank

$2n - 2$ over N . Levi flatness at $p \in N$ is equivalent to $[X, Y](p) \in HN$ whenever X, Y are local sections of HN . Thus $p \in N$ is not Levi flat if HN defines a contact structure near p ; the converse is also true when $\dim N = 3$. As real vector bundles, $HN, H^{1,0}N$, and $H^{0,1}N$ are isomorphic, for example $H^{1,0}N \ni X \mapsto \operatorname{Re} X \in HN$ establishing an isomorphism between the first two. HN comes equipped with an endomorphism J that maps $\operatorname{Re} X \in HN$ to $-\operatorname{Im} X \in HN$ for any $X \in H^{1,0}N$. Obviously, $J^2 = -\operatorname{id}$. In particular, we see that the bundle HN has a canonical orientation.

By a transverse loop Γ in N we shall mean an equivalence class of smooth immersions $f: S^1 \rightarrow N$ such that for every $t \in S^1, f(S^1)$ is transverse to $H_{f(t)}N$. Two immersions f_1, f_2 are equivalent if $f_1 = f_2 \circ \varphi$, with $\varphi: S^1 \rightarrow S^1$ a smooth orientation preserving diffeomorphism. Somewhat abusively, we shall also denote by Γ the image $f(S^1) \subset N$ of an f representing $\Gamma = [f]$. Denote the set of transverse loops in N by \mathfrak{N} .

\mathfrak{N} is a topological space, indeed a Fréchet manifold. To define these structures, endow N with a Riemannian metric. It will be convenient to assume the metric is complete, although this is not essential; the structure on \mathfrak{N} will be independent of the metric.

Let $\Gamma \subset N$ be a transverse loop in N , represented by $f: S^1 \rightarrow N$. Denote by \exp the exponential map $f^*HN \rightarrow N$. This is an immersion when restricted to a sufficiently small neighborhood of the zero section. Again, f^*HN inherits a metric and a connection from the metric on N , so we can introduce C^k norms $|s|_k$ for sections $s \in C^\infty(f^*HN)$. Given now a small positive ϵ and a positive integer k , put

$$(3.1) \quad \mathfrak{U}(f, k, \epsilon) = \{[\exp \circ s] : s \in C^\infty(f^*HN), |s|_k < \epsilon\}.$$

By declaring the sets (3.1) a neighborhood basis of Γ we obtain a topology on \mathfrak{N} . Again, there is an obvious way to identify $\mathfrak{U}(f, k, \epsilon)$ with open sets in Fréchet spaces (namely $C^\infty(f^*HN) \cong$ space of smooth mappings $S^1 \rightarrow \mathbb{R}^{2n-2}$), and this identification turns \mathfrak{N} into a Fréchet manifold. The tangent space $T_\Gamma\mathfrak{N}$ is simply $C^\infty(f^*HN)$.

We can also introduce an endomorphism \mathfrak{J} of $T_\Gamma\mathfrak{N}$ by $\mathfrak{J}s_1 = s_2$ if $s_1, s_2 \in C^\infty(f^*HN)$ are such that $Js_1(t) = s_2(t)$ for every point $t \in S^1$. Then $\mathfrak{J}^2 = -\operatorname{id}$, so \mathfrak{J} defines an almost complex structure. The $\pm i$ eigenspaces of \mathfrak{J} determine a splitting $C \otimes T\mathfrak{N} = T^{1,0}\mathfrak{N} \oplus T^{0,1}\mathfrak{N}$, and so again we have the notion of holomorphicity of a C^1 mapping from an open set $\mathfrak{U} \subset \mathfrak{N}$ into a, say, Fréchet space, and also the notion of

holomorphicity of a C^1 mapping of an (almost) complex manifold into \mathfrak{N} . For later use let us record the fact that for a loop $\Gamma = [f] \in \mathfrak{N}$ tangent vectors $\mathfrak{X} \in T^{1,0}\mathfrak{N}$ can be identified with sections $s \in C^\infty(f^*H^{1,0}N)$.

At this point we could discuss formal integrability, but we would first like to exhibit examples of holomorphic mappings into and from \mathfrak{N} . For this purpose we shall assume N is a hypersurface in a complex manifold Q . Suppose furthermore that with some domain $A \subset \mathbb{C}$ we are given a holomorphic immersion $g: A \rightarrow Q$ that is transverse to N , and moreover the smooth curve $\gamma = g^{-1}(N)$ is compact and connected. In this case g restricted to γ defines a transverse loop $\Gamma \in \mathfrak{N}$. (Observe that, as a simple closed curve in \mathbb{C} , γ has a canonical orientation.) Conversely, any real analytic transverse loop $\Gamma \in \mathfrak{N}$ comes from a holomorphic immersion of some annulus $A \subset \mathbb{C}$ as above.

Assume now that we are given a complex manifold M and a holomorphic mapping $G: M \times A \rightarrow Q$ such that $G(p_0, \zeta) = g(\zeta)$ for some $p_0 \in M$ and all $\zeta \in A$. Put $g_p(\zeta) = G(p, \zeta)$. Then for $p \in M$ close to p_0 , g_p define transverse loops $\Gamma_p \subset N$ (essentially $\Gamma_p = N \cap g_p(A)$). It is straightforward to check that the map $p \mapsto \Gamma_p \in \mathfrak{N}$ is holomorphic. In particular, suppose N is a real analytic hypersurface in Q , $\Gamma = [f]$ is a real analytic transverse loop in N , and s is a real analytic section of $f^*H^{1,0}N$, which determines a tangent vector $\mathfrak{X} \in T_\Gamma^{1,0}\mathfrak{N}$. Then there is a holomorphic curve in \mathfrak{N} through Γ in direction \mathfrak{X} ; i.e., there is a neighborhood V of $0 \in \mathbb{C}$ and a holomorphic mapping $\varphi: V \rightarrow \mathfrak{N}$ with $\varphi(0) = \Gamma$ and $\varphi_*(0)(\partial/\partial\zeta) = \mathfrak{X}$. We shall see later that without the assumption of real analyticity such holomorphic curves need not exist.

Next we will produce (local) holomorphic functions on \mathfrak{N} . To this end fix a holomorphic $(1, 0)$ form α on Q .

Proposition 3.1. *The formula*

$$(3.2) \quad F(\Gamma) = \int_\Gamma \alpha$$

defines a holomorphic function $F: \mathfrak{N} \rightarrow \mathbb{C}$.

Proof. Let $\mathfrak{X} \in T_\Gamma\mathfrak{N}$ be given by a section $s \in C^\infty(f^*HN)$ with $f: S^1 \rightarrow N$ representing Γ . For brevity, put $\tau = f_*\partial/\partial t$, t denoting the coordinate on S^1 . We claim

$$(3.3) \quad \begin{aligned} \mathfrak{X}F &= \int_{S^1} (\langle d\alpha, s(t) \wedge \tau(t) \rangle dt + d\langle \alpha, s(t) \rangle) \\ &= \int_{S^1} \langle d\alpha, s(t) \wedge \tau(t) \rangle dt. \end{aligned}$$

This would prove the proposition, since then

$$\Im \mathfrak{X}F = \int_{S^1} \langle d\alpha, Js(t) \wedge \tau(t) \rangle dt = i \int_{S^1} \langle d\alpha, s(t) \wedge \tau(t) \rangle dt = i\mathfrak{X}F,$$

(for $d\alpha$ is a $(2, 0)$ form).

As to (3.3), only the first equality needs to be proved. Assume first Γ is an *embedded* loop, and construct a vector field X on a neighborhood of $\Gamma \subset N$ such that $X_{f(t)} = s(t)$. Denote by h_θ ($|\theta| < \epsilon$) the local flow of X . Then

$$\begin{aligned} \mathfrak{X}F &= \left. \frac{d}{d\theta} \right|_{\theta=0} \int_{h_\theta(\Gamma)} \alpha = \left. \frac{d}{d\theta} \right|_{\theta=0} \int_{\Gamma} h_\theta^* \alpha \\ &= \int_{\Gamma} L_X \alpha = \int_{\Gamma} (X \lrcorner d\alpha + d(X \lrcorner \alpha)), \end{aligned}$$

which is equivalent to (3.3). If Γ is not embedded, we can represent it as a union of embedded arcs $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_m$; applying the above argument to the subarcs Γ_j , we obtain (3.3) for an arbitrary immersed loop.

Remark 3.2. There is an alternative way to reduce the case of a nonembedded $\Gamma = [f]$ to embedded loops. Endow Q with a (complete) Riemannian metric, and let $\nu = f^*TQ \ominus TS^1$ denote the normal bundle of f . Using the exponential map from ν to Q , we can pull back the complex structure of Q to a neighborhood $\tilde{Q} \subset \nu$ of the zero section. We can also pull back α to get a holomorphic $(1, 0)$ form $\tilde{\alpha}$ on \tilde{Q} . As a result we can work on the complex manifold \tilde{Q} with embedded loops. This argument also shows that even if α is a multivalued holomorphic $(1, 0)$ form, formula (3.2) (understood as $F([f]) = \int_{S^1} f_* \alpha$) still defines a holomorphic function on the open set of those loops $\Gamma = [f]$ for which $f_* \alpha$ (as a form on f^*TQ) can be made single valued.

It is clear that in Proposition 3.1 only properties of $\alpha|_N$ matter. In particular, (3.2) defines a holomorphic function F if $\alpha = u dv$ with u, v smooth CR functions on N . Putting together this construction of holomorphic functions on \mathfrak{N} with our previous construction of holomorphic mappings into \mathfrak{N} , we see that for certain families $\{\gamma_\zeta\}$ of closed curves in N , parametrized by points ζ in a complex manifold M , the integrals $\int_{\gamma_\zeta} u dv$ depend holomorphically on ζ , if u, v are smooth CR functions. This circumstance has been known and exploited for a long time, first by Lewy, and then by others; see [1], [8], [14], [21]; the idea can be traced back to F. John; see [9].

Formula (3.2) can be thought of as an infinite dimensional Radon transformation. It follows that F will satisfy an infinite system of second order partial differential equations. Consequently (3.2) cannot describe all holomorphic functions on \mathfrak{N} . More holomorphic functions can be constructed

by a generalization of (3.2) in the spirit of Chen's iterated integrals [5].

For this purpose choose a positive integer k and consider those k -tuples (t_1, t_2, \dots, t_k) of distinct points in S^1 that follow each other according to the orientation of S^1 . The set of these k -tuples will be denoted $\Delta_k \subset (S^1)^k$; thus Δ_k is a generic orbit of the diagonal action of $\text{Diff } S^1$ on $(S^1)^k$. In particular, $\Delta_1 = S^1$, $\Delta_2 = S^1 \times S^1 \setminus \text{diagonal}$, but when $k > 2$, $(S^1)^k \setminus \Delta_k$ has nonempty interior. Let α be a holomorphic $(k, 0)$ form on $Q \times \dots \times Q = Q^k$ (k factors). If $f : S^1 \rightarrow Q$ is any mapping, define $f_k : (S^1)^k \rightarrow Q^k$ by $f_k(t_1, \dots, t_k) = (f(t_1), \dots, f(t_k))$. Then we can construct a holomorphic function F on \mathfrak{N} by the formula

$$F(\Gamma) = \int_{\Delta_k} f_k^* \alpha, \quad \Gamma = [f].$$

Finite sums of functions thus constructed form an algebra; we conjecture that locally on \mathfrak{N} this algebra is dense in the space of holomorphic functions.

There is another way of generalizing the construction of holomorphic functions on \mathfrak{N} as given in Proposition 3.1. This is based on a remark by Brylinski to the effect that an integral like the one in (3.2) should be thought of as a holonomy. Accordingly, instead of a holomorphic $(1, 0)$ form on Q , consider a holomorphic principal G -bundle $P \rightarrow Q$ with a holomorphic connection. Here G is a complex Lie group. If $f : S^1 \rightarrow N$ is a transverse immersion, horizontal lift along f defines a holonomy $g(f) \in G$. If $\varphi \in \text{Diff } S^1$, $g(f \circ \varphi) = g(f)$, so g in fact descends to a group valued function $F : \mathfrak{N} \rightarrow G$; and it is not hard to check that F is holomorphic.

As a matter of fact, this construction can be made more intrinsic to N , if, instead of a holomorphic principal bundle over Q , one takes a CR principal G -bundle over N (G still complex Lie group) with a CR connection. Holonomy again will define a holomorphic function $F : \mathfrak{N} \rightarrow G$. In this fashion one can even construct holomorphic functions on loop spaces of nonembeddable CR manifolds, at least in principle. The difficulty of course lies in finding CR bundles with CR connections over N . We hope to return to this question in a later publication.

4. Weak integrability

Again we shall assume that our CR manifold N is a hypersurface in a complex manifold Q . We shall prove that the almost complex structure

\mathfrak{J} on \mathfrak{N} is locally integrable in a weak sense, hence it is also formally integrable.

Theorem 4.1. *Any $\Gamma \in \mathfrak{N}$ has a neighborhood \mathfrak{N}_0 such that for any nonzero $\mathfrak{X} \in T_{\Gamma}^* \mathfrak{N}$ there is a holomorphic function F on \mathfrak{N}_0 with $\mathfrak{X}F \neq 0$.*

We will prove this theorem after some preparation.

Proposition 4.2. *Let A be a doubly connected Riemann surface, $\gamma \subset A$ a smooth Jordan curve, not null homotopic. Given a continuous 1-form φ on γ such that $\int_{\gamma} \varphi = 0$ and a positive ϵ , there exists a holomorphic function h on A such that $|dh - \varphi| < \epsilon$ on γ . Here the uniform norm $\| \cdot \|$ of a 1-form is measured using some fixed Riemannian metric on A .*

Proof. By the uniformization theorem we can assume that $A = \{ \zeta \in \mathbb{C} : r < |\zeta| < R \}$. Write $\varphi = \varphi_1 d\zeta$, with φ_1 a continuous function on γ . We can uniformly approximate φ_1 by Laurent polynomials of form $\psi_1(\zeta) = \sum_{-k}^k a_j \zeta^j$ (see, e.g., [15]). Here

$$a_{-1} = \frac{1}{2\pi i} \int_{\gamma} \psi_1 d\zeta = \frac{1}{2\pi i} \int_{\gamma} (\psi_1 - \varphi_1) d\zeta,$$

which is small, so that $\psi_1 - a_{-1}\zeta^{-1}$ is also close to φ_1 on γ . Hence $h(\zeta) = \int (\psi_1(\zeta) - a_{-1}\zeta^{-1}) d\zeta$ will do.

Proposition 4.3. *Any smooth Jordan curve $\gamma \subset \mathbb{C}^m$ has a neighborhood U with the following property. Given a continuous 1-form $\varphi = \sum_1^m \varphi_j dz_j$ along γ such that $\int_{\gamma} \varphi_j dz_j = 0$ ($j = 1, \dots, m$), and $\epsilon > 0$, there is a holomorphic function h on U such that $|dh - \varphi| < \epsilon$ along γ .*

Proof. By a generic linear change of coordinates we can achieve that the coordinate projections $\pi_j : \mathbb{C}^m \rightarrow \mathbb{C}$ restrict to immersions on γ . Then the fibers of π_j are transverse to γ , so we can find a neighborhood U of γ such that any connected component of $U \cap \pi_j^{-1}(\zeta)$ intersects γ at one point at most ($j = 1, \dots, m$). Pick now a j . Let A_j denote the leaf space of the holomorphic foliation of U determined by the fibers of π_j . This is a Riemann surface (spread over \mathbb{C}). After shrinking U , we can assume A_j is doubly connected. With $\sigma_j : U \rightarrow A_j$ denoting the canonical projection, $\sigma_j(\gamma) \subset A_j$ is a smooth Jordan curve, not null homotopic. By Proposition 4.2 there is a holomorphic function h_j on A_j such that dh_j approximates (the push forward to $\sigma_j(\gamma)$ of) $\varphi_j dz_j$. Then $h = \sum h_j \circ \sigma_j$ will do.

Proof of Theorem 4.1. Assume first that Γ is an embedded loop represented by $f : S^1 \rightarrow N$. With an arbitrary Riemannian metric on Q put $r(q) = \text{dist}^2(q, \Gamma)$ ($q \in Q$). This is a strictly plurisubharmonic function in a neighborhood of $\Gamma \subset Q$, whence for $\epsilon > 0$ sufficiently small

$Q_1 = \{q \in Q : r(q) < \epsilon\}$ is a Stein manifold and so embeds into some Euclidean space \mathbb{C}^m . (For these matters, see [2], [6], [17]). Accordingly, we shall think of Q_1 and $N_1 = N \cap Q_1$ as submanifolds of \mathbb{C}^m . In addition we can assume that the coordinate projections restrict to immersions on Γ .

The vector $\mathfrak{X} \in T_\Gamma \mathfrak{N}$ is represented by a section $s \in C^\infty(f^*HN)$, $[f] = \Gamma$. There is an interval on which s does not vanish, which we can take to be $[\pi/2, 3\pi/2] \subset S^1$ (now we think of S^1 as $\mathbb{R} \bmod 2\pi$). Choose a real valued function $\rho \in C^\infty(S^1)$, $\text{supp } \rho \subset [\pi/2, 3\pi/2]$, $\rho(\pi) = 1$. Put $\tau(t) = f_*(t)\partial/\partial t \in T\Gamma$ as before. Since for $t \in [\pi/2, 3\pi/2]$, $\tau(t), s(t) \in T_{f(t)}\mathbb{C}^m$ are independent, we can construct smooth $(1, 0)$ forms $\varphi = \sum \varphi_j dz_j$, $\psi = \sum \psi_j dz_j$ on \mathbb{C}^m so that we have

$$(4.1) \quad \langle \varphi(f(t)), s(t) \rangle = \rho(t), \quad \langle \varphi(f(t)), \tau(t) \rangle = 0 \quad (0 \leq t \leq 3\pi/2),$$

$$(4.2) \quad \langle \psi(f(t)), s(t) \rangle = 0, \quad \langle \psi(f(t)), \tau(t) \rangle = \rho(t) \quad (\pi/2 \leq t \leq 2\pi).$$

Conditions (4.1), (4.2) do not restrict $\varphi(f(t)), \psi(f(t))$ for $3\pi/2 < t < 2\pi$, resp. $0 < t < \pi/2$. We can use this freedom to arrange that in addition to (4.1), (4.2) also

$$\int_\Gamma \varphi_j dz_j = \int_\Gamma \psi_j dz_j = 0 \quad (j = 1, \dots, m)$$

hold.

By virtue of Proposition 4.3 on Γ we can approximate φ (resp. ψ) by du (resp. dv) with u, v holomorphic functions on a fixed neighborhood $U \subset \mathbb{C}^m$ of Γ . Put $\alpha = u dv$. If $\gamma \in \mathfrak{N}$ is sufficiently close to Γ , then its image in N (also denoted γ) lies entirely in U , so we can define a holomorphic function F in a neighborhood \mathfrak{N}_0 of $\Gamma \in \mathfrak{N}$ by $F(\gamma) = \int_\gamma \alpha$, as in Proposition 3.1.

According to (3.3), we have

$$\mathfrak{X}F = \int_{S^1} (\langle du, s(t) \rangle \langle dv, \tau(t) \rangle - \langle du, \tau(t) \rangle \langle dv, s(t) \rangle) dt,$$

which, by appropriate choice of u, v can be made arbitrarily close to

$$\begin{aligned} & \int_{S^1} (\langle \varphi(f(t)), s(t) \rangle \langle \psi(f(t)), \tau(t) \rangle - \langle \varphi(f(t)), \tau(t) \rangle \langle \psi(f(t)), s(t) \rangle) dt \\ & = \int_{S^1} \rho^2(t) dt > 0. \end{aligned}$$

Thus, with an appropriate choice of u, v $\mathfrak{X}F \neq 0$.

The above argument can be enhanced to take care of general immersed loops, as explained in Remark 3.2.

Corollary 4.4. *The almost complex structure \mathfrak{J} on \mathfrak{N} is formally integrable.*

Proof. Let $\mathfrak{Y}, \mathfrak{Z}$ be local sections of $T^{1,0}\mathfrak{N}$, and write $[\mathfrak{Y}, \mathfrak{Z}] = \mathfrak{W} + \mathfrak{X}$ with \mathfrak{W} a section of $T^{1,0}\mathfrak{N}$, \mathfrak{X} a section of $T\mathfrak{N}$. For any local holomorphic function F we have

$$\mathfrak{X}F = \mathfrak{Y}\bar{\mathfrak{Z}}F - \bar{\mathfrak{Z}}\mathfrak{Y}F - \mathfrak{W}F = 0.$$

Hence $\mathfrak{X} = 0$ by Theorem 4.1.

Remark 4.5. With a little more work Theorem 4.1 can be strengthened as follows. Any $\Gamma \in \mathfrak{N}$ has a neighborhood \mathfrak{N}_0 such that, in addition to Theorem 4.1, for any $\Gamma_1, \Gamma_2 \in \mathfrak{N}_0$ ($\Gamma_1 \neq \Gamma_2$) there is a holomorphic $F: \mathfrak{N}_0 \rightarrow \mathbb{C}$ with $F(\Gamma_1) \neq F(\Gamma_2)$.

Remark 4.6. We do not know if Theorem 4.1 remains true without the assumption that N can be embedded in a complex manifold. On the other hand Corollary 4.4 is true for any CR manifold N , as can be checked by direct computations (or by approximating the CR structure of N , locally near a loop, by embeddable CR structures).

5. Failure of strong integrability

Again assume N is a hypersurface in a complex manifold Q , $\dim_{\mathbb{C}} Q = n$.

Theorem 5.1. *Suppose N has a non-Levi flat point. Then \mathfrak{N} is not locally biholomorphic to open sets in Fréchet spaces.*

It is not hard to show that when N is everywhere Levi flat, it is locally CR equivalent to $S^1 \times \mathbb{C}^{n-1}$ and hence \mathfrak{N} is locally biholomorphic to $C^\infty(S^1 \rightarrow \mathbb{C}^{n-1})$. We remark that in this case (and only in this case, cf. Proposition 5.5) \mathfrak{N} can be empty.

The proof of Theorem 5.1 is based on the following observation, which is true even when N is not embedded in a complex manifold.

Proposition 5.2. *Let $D \subset \mathbb{C}^{n-1}$ be an open set, and $\Phi: D \rightarrow \mathfrak{N}$ a holomorphic mapping (of class C^1). Suppose that for points q in some open set $U \subset N$ there is a unique $\zeta = u(q) \in D$ such that the (range of the) loop $\Phi(\zeta)$ contains q . Assume furthermore that $u: U \rightarrow \mathbb{C}^{n-1}$ is of class C^1 . Then u is a CR function and a submersion.*

Proof. Let $q_0 \in U$, $\zeta_0 = u(q_0)$. After possibly shrinking D , we can construct a C^1 mapping $f: D \times S^1 \rightarrow N$ such that the mappings $f_\zeta = f(\zeta, \cdot)$ are smooth immersions and represent the loops $\Phi(\zeta)$. Put also

$f^t = f(\cdot, t)$ ($t \in S^1$). Let t_0 be such that $f(\zeta_0, t_0) = q_0$. For (ζ, t) close to (ζ_0, t_0) we have

$$(5.1) \quad u(f(\zeta, t)) = \zeta,$$

whence $\text{rk } u_* = 2n - 2$, so u is a submersion. By reparametrization we can arrange that $f_*^{t_0}(T_{\zeta_0} D) \subset H_{q_0} N$. Then (5.1) implies that $f_*^{t_0}|_{T_{\zeta_0} D}$ and $u_*|_{H_{q_0} N}$ are inverses of one another. Since by virtue of the holomorphicity of Φ the former intertwines the almost complex endomorphisms J_D of D and J of N , the same holds for the latter, whence u indeed satisfies the tangential Cauchy-Riemann equations.

In the following proposition we shall think of S^1 as $\mathbb{R} \bmod 2\pi$. Henceforward we shall need that N is a hypersurface in a complex manifold Q .

Proposition 5.3. *Suppose $f : S^1 \rightarrow N$ is a smooth immersion such that for some $t_0 \in S^1$ $p = f(t_0) \in N$ is a non-Levi flat point, and $s_1, \dots, s_{n-1} \in C^\infty(f^* H^{1,0} N)$ are such that $s_1(t_0), \dots, s_{n-1}(t_0)$ form a basis of $H_f^{1,0} N$. f defines a loop $\Gamma \in \mathfrak{N}$ and s_1, \dots, s_{n-1} determine tangent vectors $\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1} \in T_\Gamma^{1,0} \mathfrak{N}$. If there is a holomorphic mapping $\Phi : D \rightarrow \mathfrak{N}$ of class C^1 , with D some neighborhood of $0 \in \mathbb{C}^{n-1}$, such that $\Phi(0) = \Gamma$, and $\Phi_*(0)\partial/\partial\zeta_j = X_j$ ($j = 1, \dots, n - 1$), then*

(a) *there is a Riemann surface $\Sigma \subset Q$ with C^1 boundary such that $\partial\Sigma$ contains an arc $f(t_0 - \epsilon, t_0 + \epsilon)$ of Γ , for some $\epsilon > 0$;*

(b) *there are continuous sections X_j of $T^{1,0} Q|_\Sigma$, holomorphic on $\text{int } \Sigma$, such that*

$$(5.2) \quad X_j(f(t)) \equiv s_j(t) \pmod{T_{f(t)}^{1,0} \Sigma}, \quad t_0 - \epsilon < t < t_0 + \epsilon,$$

$j = 1, 2, \dots, n - 1$.

Proof. (a) Assume f is an embedding (we already know how to lift merely immersed loops to embedded ones). The implicit function theorem (on N) implies that for q in a neighborhood $U \subset N$ of p there is a unique $\zeta = u(q)$ in a neighborhood $D_0 \subset D$ of 0 such that the loop $\Phi(\zeta)$ passes through q ; furthermore $u : U \rightarrow \mathbb{C}^{n-1}$ is of class C^1 . By Proposition 5.2 u is CR, and so by Lewy's theorem there are a neighborhood $G \subset Q$ of p cut in two by N , and a holomorphic mapping \tilde{u} on one of the components, say G_1 , C^1 on $G_1 \cup (N \cap G)$, such that $\tilde{u} = u$ on $N \cap G$. Obviously (if G is sufficiently small) \tilde{u} is still a submersion, whence $\Sigma = \tilde{u}^{-1}(0) \subset Q$ is a Riemann surface with C^1 boundary. The boundary near $p \in \Sigma$ agrees with an arc $f(t_0 - \epsilon, t_0 + \epsilon)$ of the loop Γ .

(b) Fix j . For $q \in \Sigma$ the vectors in $T_q^{1,0}Q$ that $u_*(q)$ maps to $\partial/\partial\zeta_j \in T_0^{1,0}\mathbb{C}^{n-1}$ form an equivalence class in $T_q^{1,0}Q/T_q^{1,0}\Sigma$. From (5.1) one can read off that when $q = \tilde{f}(t) \in \Sigma \cap N$, $s_j(t)$ is in this equivalence class. When we let q vary on Σ , these equivalence classes define a continuous section of the normal bundle of Σ in Q , holomorphic on $\text{int}\Sigma$. This section can be lifted to a continuous section X_j of $T^{1,0}Q|_\Sigma$, holomorphic on $\text{int}\Sigma$. Clearly X_j has the required property.

Remark 5.4. When N is strictly pseudoconvex, a converse to Proposition 5.3 is also true. Suppose $f: S^1 \rightarrow N$ defines a loop $\Gamma \in \mathfrak{N}$ and f has a holomorphic continuation \tilde{f} to $\Pi_\epsilon = \{t \in \mathbb{C} : 0 < \text{Im } t < \epsilon\}$ such that $\tilde{f}(\Pi_\epsilon)$ lies on the pseudoconvex side of N . Let $\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1} \in T^{1,0}\mathfrak{N}$ be determined by sections $s_1, \dots, s_{n-1} \in C^\infty(f^*H^{1,0}N)$ that have, mod $\tilde{f}_*T^{1,0}\Pi_\epsilon$, holomorphic continuations to Π_ϵ (as sections of $\tilde{f}_*T^{1,0}Q$). Then there is a smooth holomorphic mapping $\Phi: D \rightarrow \mathfrak{N}$ with $D \subset \mathbb{C}^{n-1}$ a neighborhood of 0 such that $\Phi(0) = \Gamma$ and $\Phi_*(0)(\partial/\partial\zeta_j) = \mathfrak{x}_j$.

Indeed (when Γ is an embedded loop), the loops $\Phi(\zeta)$ can be obtained as fibers $N \cap \tilde{u}^{-1}(\zeta)$, with \tilde{u} a suitable holomorphic submersion of the pseudoconvex side of N , into \mathbb{C}^{n-1} , which is smooth up to N .

To prove Theorem 5.1, we shall need one more result.

Proposition 5.5. *Suppose N has a non-Levi flat point p_0 . Then there is a loop $[f] = \Gamma \in \mathfrak{N}$ and sections $s_j \in C^\infty(f^*H^{1,0}N)$ defining tangent vectors $\mathfrak{x}_j \in T_\Gamma^{1,0}\mathfrak{N}$ ($j = 1, \dots, n-1$) such that for every $t \in S^1$, $f(t) \in N$ is not Levi flat, $s_1(t), \dots, s_{n-1}(t)$ are independent and for no $t_0 \in S^1$ and $\epsilon > 0$ are (a) and (b) of Proposition 5.3 simultaneously satisfied.*

Proof. First transversely intersect N with a two-dimensional complex manifold $Q_1 \subset Q$, $p_0 \in Q_1$, so that $N \cap Q_1 = N_1$ still be non-Levi flat at p_0 . Since $\dim N_1 = 3$, this in fact means p_0 is a strictly pseudoconvex point of N_1 . Therefore by shrinking N and Q we can assume that $N \subset \mathbb{C}^n$,

$$N_1 = \{z \in N : z_3 = \dots = z_n = 0\}$$

is a strictly convex hypersurface in $\mathbb{C}^2 \times \{0\}$ (see, e.g., [11]), and indeed N_1 is given by equations

$$\text{Im } z_1 = C(z_1, \text{Re } z_2), \quad z_3 = \dots = z_n = 0,$$

where C is a smooth, nonnegative, strictly convex function defined on some ball $\{(z_1, x_2) : |z_1|^2 + x_2^2 < r\}$, $C(0) = 0$, $\text{grad } C(0) = 0$.

distinct), the cross ratio $\mu^\# = (\mu, f^*\omega, \nu^{*0,1}, \nu^{*1,0})$ defines a smooth function $\mu^\# : S^1 \rightarrow \mathbb{C} \cup \{\infty\}$, and the correspondence $\mu \mapsto \mu^\#$ is one-to-one and onto. Furthermore, $\mu^\#$ has finite values if $\mu \neq \nu^{*1,0}$; it is zero when $\mu = \nu^{*0,1}$.

Similarly, if $h : S^1 \rightarrow M$ is close to f , and μ is a section of the pullback bundle B_h , we can define $\mu^\# = (\mu, h^*\omega, \nu_h^{*0,1}, \nu_h^{*1,0})$.

To construct the diffeomorphism Δ , let $s \in C^\infty(\nu)$ determine an immersion $h = \exp \circ s : S^1 \rightarrow M$ close to f (cf. (1.1)), and let $\rho \in C^\infty(S^1)$ be arbitrary. There is a unique smooth section μ of B_h such that $\mu^\# = \rho$. If $\rho \equiv 0$, then $\mu : S^1 \rightarrow N$ becomes a negative Legendrean immersion, hence it is transverse to the CR structure. It follows that for ρ in a neighborhood \mathfrak{M} of $0 \in C^\infty(S^1)$ μ is still transverse, hence defines a transverse loop $[\mu] \in \mathfrak{N}$. Put $\Delta([h], \rho) = [\mu]$. It is a simple exercise to check that Δ is indeed a locally defined diffeomorphism $\mathfrak{M} \times C^\infty(S^1) \rightarrow \mathfrak{N}$, and its restriction to $\mathfrak{M} \times \{0\}$ is Θ .

Theorem 9.5. Θ is holomorphic.

Proof. Let $\Gamma \in \mathfrak{M}$ be represented by $f : S^1 \rightarrow M$; then $\tilde{\Gamma} = \Theta(\Gamma)$ is represented by the Legendrean lift $\tilde{f} : S^1 \rightarrow N$ of f . Since $T_{\tilde{\Gamma}}^{0,1}\mathfrak{M}$ (resp. $T_{\tilde{\Gamma}}^{0,1}\mathfrak{N}$) can be identified with $C^\infty(\nu_f^{0,1})$ (resp. $C^\infty(\tilde{f}^*H^{0,1}N)$), Proposition 9.2 implies $\Pi_* T_{\tilde{\Gamma}}^{0,1}\mathfrak{N} = T_{\tilde{\Gamma}}^{0,1}\mathfrak{M}$, i.e., $\Pi_*(\tilde{\Gamma})$ intertwines the almost complex structure tensors $\mathfrak{J}_{\mathfrak{N}}$, $\mathfrak{J}_{\mathfrak{M}}$. This being true for any $\tilde{\Gamma} \in \mathfrak{L}$, it follows that \mathfrak{L} is a complex submanifold (in the sense that $T\mathfrak{L}$ is $\mathfrak{J}_{\mathfrak{N}}$ invariant), and $\Pi|_{\mathfrak{L}}$ and its inverse, Θ , are holomorphic.

Corollary 9.6. \mathfrak{M} and \mathfrak{L} are biholomorphic via a smooth biholomorphism. If (M, g) is real analytic, then both are locally integrable in a weak sense. For example, any $\Gamma \in \mathfrak{M}$ has a neighborhood $\mathfrak{U} \subset \mathfrak{M}$ such that for any nonzero $\mathfrak{X} \in T_{\Gamma}\mathfrak{M}$ there is a holomorphic $F : \mathfrak{U} \rightarrow \mathbb{C}$ with $\mathfrak{X}F \neq 0$.

This clearly follows from Theorem 4.1 and the fact that the twistor CR manifold of an analytic Riemannian three-manifold is itself real analytic, and hence embeddable into a complex manifold as a hypersurface.

10. Failure of strong integrability

Theorem 10.1. Assume that (M, g) is real analytic. Then no open subset $\mathfrak{U} \subset \mathfrak{M}$, $\mathfrak{U} \neq \emptyset$ is biholomorphic to an open subset of a Fréchet space.

Instead of \mathfrak{M} we can work with the manifold \mathfrak{L} of transverse negative Legendrean loops in N . We shall need some information about the

tangent bundle of \mathcal{L} . In the following Propositions we do not assume (M, g) to be analytic.

Proposition 10.2. *If $[f]=\Gamma \in \mathcal{L}$, and $T_\Gamma \mathfrak{N}$ is identified with $C^\infty(f^*HN)$, then $T_\Gamma \mathcal{L}$ corresponds to the subset $\Lambda_f \subset C^\infty(f^*HN)$ consisting of smooth Legendrean vector fields along f (in the sense of Definition 8.2).*

Proof. If $f_\epsilon : S^1 \rightarrow N$ is a one-parameter family of transverse negative Legendrean immersions such that $f_0 = f$, and $\sigma = df_\epsilon/d\epsilon|_{\epsilon=0} \in C^\infty(f^*HN)$ represents a tangent vector in $T_\Gamma \mathcal{L}$, then by definition σ is a Legendrean vector field. Conversely, if $\sigma \in C^\infty(f^*HN)$ is Legendrean, then its image under the projection $\pi : N \rightarrow M$ is

$$\pi_*\sigma = \tau \in C^\infty(\nu_{\pi \circ f}).$$

Let φ_ϵ be a family of immersions $S^1 \rightarrow M$ such that $\varphi_0 = \pi \circ f$, and $d\varphi_\epsilon/d\epsilon|_{\epsilon=0} = \tau$. Then the velocity vector of the Legendrean lifts $f_\epsilon = \tilde{\varphi}_\epsilon : df_\epsilon/d\epsilon|_{\epsilon=0}$ is σ , i.e., σ corresponds to a vector in $T_\Gamma \mathcal{L}$.

Corollary 10.3. *Λ_f is a vector subspace of $C^\infty(f^*HN)$. If $\sigma \in \Lambda_f$, then $J\sigma \in \Lambda_f$. Further, complexified tangent vectors in $\mathbb{C} \otimes T_\Gamma \mathcal{L}$ correspond to elements of $\mathbb{C} \otimes \Lambda_f$, and tangent vectors in $T_\Gamma^{1,0} \mathcal{L}$ (resp. $T_\Gamma^{0,1} \mathcal{L}$) correspond to vector fields of form $\sigma - iJ\sigma$ (resp. $\sigma + iJ\sigma$), $\sigma \in \Lambda_f$.*

We shall denote the space of vector fields of form $\sigma - iJ\sigma$ ($\sigma \in \Lambda_f$) by $\Lambda_f^{1,0}$.

Proposition 10.4. *Given $[f]=\Gamma \in \mathcal{L}$, $t_0 \in S^1$, and $v \in \mathbb{C} \otimes H_{f(t_0)}N$, there is a (complexified) Legendrean vector field $\sigma \in \mathbb{C} \otimes \Lambda_f$ such that $\sigma(t_0) = v$.*

Proof. It suffices to treat the case $v \in H_{f(t_0)}N$. Issue a smooth curve p_ϵ in N with $p_0 = f(t_0)$, $dp_\epsilon/d\epsilon|_{\epsilon=0} = v$. There is a smooth family of immersions $\varphi_\epsilon : S^1 \rightarrow M$ such that $\varphi_0 = \pi \circ f$, $\varphi_\epsilon(t_0) = \pi(p_\epsilon)$, and $\varphi_{\epsilon*}(t_0)\partial/\partial t \in T_{\pi(p_\epsilon)}M$ is in the kernel of $p_\epsilon \in \mathbb{C} \otimes T_{\pi(p_\epsilon)}^*M$, for every ϵ . Denote the (Legendrean) lift of φ_ϵ by f_ϵ . Then $f_0 = f$, and $\sigma = df_\epsilon/d\epsilon|_{\epsilon=0} \in \Lambda_f$ does it.

Proof of Theorem 10.1. If (M, g) is analytic, then so is $(N, H^{0,1}N)$, so that this latter can be embedded as a hypersurface in a complex manifold Q . Given an open set $\mathfrak{U} \subset \mathfrak{M}$, $\mathfrak{U} \neq \emptyset$, fix a nonanalytic embedded loop $\Gamma_0 \in \mathfrak{U}$. Then $\Theta(\Gamma_0) = \Gamma = [f]$ is nonanalytic either. Let $t_0 \in S^1$ be such that for no $\epsilon > 0$ is $f(t_0 - \epsilon, t_0 + \epsilon)$ an analytic arc. Choose two independent vectors $v_1, v_2 \in H_{f(t_0)}^{1,0}N$, and $\sigma_1, \sigma_2 \in \Lambda_f^{1,0}$ such that $\sigma_j(t_0) = v_j$ ($j = 1, 2$); cf. Proposition 10.4. σ_1, σ_2 correspond to tangent vectors $X_1, X_2 \in T_\Gamma^{1,0} \mathcal{L}$.

If \mathfrak{U} were biholomorphic to an open subset of a Fréchet space, then so would be $\mathfrak{W} = \Theta(\mathfrak{U})$, which is an open subset of \mathfrak{L} . In this case there would exist an open neighborhood D of $0 \in \mathbb{C}^2$ and a holomorphic mapping $\Phi : D \rightarrow \mathfrak{W} \subset \mathfrak{N}$ with $\Phi(0) = \Gamma$, $\Phi_*(0)\partial/\partial\zeta_j = X_j$ ($j = 1, 2$). As in the proof of Proposition 5.3, we could find a neighborhood $U \subset N$ of $f(t_0) \in N$ and a CR submersion $u : U \rightarrow \mathbb{C}^2$ (of class C^1) such that $\{u = 0\}$ agrees with the portion of the (range of the) loop Γ in U . As said before, the Levi form of N has signature $(+, -)$, whence any CR function on U extends to a holomorphic function on some neighborhood $G \subset Q$ of $f(t_0)$. In particular u extends to a holomorphic submersion $\tilde{u} : G \rightarrow \mathbb{C}^2$. Then $\{\tilde{u} = 0\} = \Sigma$ is a Riemann surface whose transverse intersection with N contains an arc $f(t_0 - \epsilon, t_0 + \epsilon)$. But this is a contradiction, since $N \subset Q$ is an analytic hypersurface, so $\Sigma \cap N$ is an analytic arc. This contradiction proves the theorem.

Similarly as in §6 we can prove a slightly weaker statement for smooth Riemannian manifolds; details will be left to the reader.

Theorem 10.5. *If (M, g) is a smooth Riemannian manifold, then no open subset $\mathfrak{U} \subset \mathfrak{M}$, $\mathfrak{U} \neq \emptyset$, is smoothly biholomorphic to an open subset of a Fréchet space.*

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